

Mass generation and symmetry breaking in Chern-Simons supergravity

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We argue that the quartic fermionic potential of five-dimensional Chern-Simons supergravity induces spontaneous symmetry breaking, in a phenomenon bearing a close connection with the Nambu–Jona-Lasinio model.

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An attractive feature of five-dimensional Chern-Simons supergravity based on the Lie algebra $SU(2,2|N)$ is that it contains $SU(N)$ gauge fields, fermions and gravity. The gauge field $\Omega \in SU(2,2|N)$ has the matrix presentation

$$\Omega = \begin{pmatrix} W & \psi^j \\ \bar{\psi}^j & \mathcal{A}^{ij} \end{pmatrix}$$

where $W \in U(2,2)$, ψ_i are N Dirac spinors, and $\mathcal{A}^{ij}$ are $U(N)$ gauge fields. The action, first considered by Chamseddine [1], is

$$I[\Omega] = -ik \int \text{STr} \left(\frac{1}{3} d\Omega d\Omega \Omega + \frac{1}{2} d\Omega \Omega^3 + \frac{1}{5} \Omega^5 \right) \\ = I[W] + I_W[\psi] \quad (1)$$

where $I[W]$ is a Chern-Simons theory by itself. Since we shall be interested in the gravitational and fermionic degrees of freedom we set $\text{Tr}(W) = \mathcal{A} = 0$ hereafter.

Note that this theory has only two (dimensionless) parameters: the number of colors N and the level k . The gravitational equations of motion are

$$\mathcal{R} \wedge \mathcal{R} = - \frac{\delta I_W[\psi]}{\delta W} \quad (2)$$

with $\mathcal{R} = dW + WW$. On the fermion ground state, it is generally taken for granted that the right hand side vanishes. The goal of this paper is to study this assumption in detail.

Chern-Simons gravity was first introduced in [2] in three dimensions. It was then pointed out [1] that the same construction can be carried over to five dimensions. Poincaré Chern-Simons supergravities were introduced in [3], and their anti-de Sitter extensions in [4]. It was conjectured in [4] and [5] that M theory may be described by eleven-dimensional Chern-Simons supergravity. This idea has recently been reexamined in [6]. A possible mechanism relating the field content of standard and Chern-Simons supergravities was discussed in [7]. The dynamical structure of higher dimensional Chern-Simons theories and their asso-

ciated current algebras was studied in [8–10]. See also [11] for other aspects of higher-dimensional Chern-Simons theories.

Let us start by recalling the meaning of the Chern-Simons level k and its relation to Planck's scale. In the purely gravitational sector, the gauge field W is expanded as

$$W = \frac{1}{2l} e^a \gamma_a + \frac{1}{4} w^{ab} \gamma_{ab} \quad (3)$$

where e^a is identified as the veilbein, and w^{ab} the spin connection. The length parameter l (AdS radius) is introduced here because W_μ has dimensions 1/length while e_μ^a , related to the spacetime metric by $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$, has dimension zero.

The next step is to find Planck's length, the small scale parameter, in terms of l and k . Expanding $I[W]$ in powers of the curvature tensor we can identify the term $(k/2l^3) \sqrt{-g} R$ and we thus find

$$k = \frac{l^3}{l_p^3} \quad (4)$$

where $l_p^3 = 8\pi G$. The Chern-Simons level k in supergravity then measures the quotient “maximum length”–“minimum length.” In the weakly coupled theory with large k both lengths are far away from each other.

Having introduced the relevant parameters in our theory we shall now set, for notational simplicity, $l = 1$. In this unities, Planck's length and all associated parameters depend only on k . For later use we only quote Planck's energy

$$E_p = \frac{1}{l_p} = k^{1/3}. \quad (5)$$

The fermionic term in Eq. (1) with $N = 1$ is [1]

$$I_W[\psi] = \int \left(2i \bar{\psi} \mathcal{R} \nabla \psi + \frac{4i}{k} \bar{\psi} \psi \bar{\psi} \nabla \psi \right) \quad (6)$$

where

$$\nabla \psi = D\psi + \frac{1}{2} e^a \gamma_a \psi$$

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$$\mathcal{R} = \frac{1}{4}(R^{ab} + e^a e^b) \gamma_{ab}. \quad (7)$$

D is the Lorentz covariant derivative with $D \wedge D = R$ and $\nabla \wedge \nabla = \mathcal{R}$. For later convenience, we have rescaled the fermions by $\psi \rightarrow (2/k)^{1/2} \psi$.

As in any supergravity theory, the free part of Eq. (6) is linearly supersymmetric around $\psi=0$. The variation of the free part of Eq. (6) under

$$\delta\psi = \nabla \epsilon$$

yields $\bar{\psi} \mathcal{R} \epsilon$ which is zero when the linear bosonic equations of motion (2) hold. The key question is whether $\psi=0$ is the true ground state of the theory or not. If this was not true, the expansion around a nonzero value $\langle \psi \rangle$ would introduce other quadratic terms coming from the potential. Then, the linear transformations would not cancel on the background (2) and the ground state would not be supersymmetric.

We shall argue here that this in fact occurs in the generic situation. In particular we prove that the fermion condensate

$$\langle \bar{\psi} \gamma^{ab} \psi \rangle \neq 0 \quad (8)$$

is different from zero. Our analysis has a close analogy with the Nambu–Jona-Lasinio model [12], although we follow the auxiliary field formalism of Gross and Neveu [13].

The occurrence of Eq. (8) follows from the following observation. Recalling the expressions (7) for ∇ , and applying the five-dimensional Fierz identity [14] it is a simple exercise to express the interaction term as

$$\frac{4i}{k} \bar{\psi} \psi \bar{\psi} \nabla \psi = \frac{1}{k} \bar{\psi} \psi \bar{\psi} D \psi - \frac{1}{4k} \epsilon_{abcde} e^a \bar{\psi} \gamma^{bc} \psi \bar{\psi} \gamma^{de} \psi.$$

Both terms are perturbations of the same order in powers of k . The second term has the structure $(\bar{\psi} \gamma^{ab} \psi)^2$, similar to that arising in the Gross-Neveu model, and we shall then focus on it. It is important to stress, however, that the first term could be relevant and it may turn on other operators like $\langle \bar{\psi} \gamma^{ab} D \psi \rangle$; we shall study this possibility elsewhere.

Keeping only the second term we rewrite the interaction introducing an auxiliary field σ^{ab} ,

$$- \frac{1}{4k} \bar{\psi} \psi \bar{\psi} \nabla \psi \rightarrow \frac{k}{4} \epsilon_{abcde} e^a \sigma^{bc} \sigma^{de} - \frac{1}{2} \epsilon_{abcde} e^a \sigma^{bc} \bar{\psi} \gamma^{de} \psi$$

whose equation of motion is

$$\sigma^{ab} = \frac{1}{k} \bar{\psi} \gamma^{ab} \psi.$$

The weak coupling, $k \rightarrow \infty$, fermionic action we consider is then

$$I_W[\psi, \sigma] = \int \left[\bar{\psi} \left(2i \mathcal{R} \nabla - \frac{\phi}{2} \right) \psi + \frac{k}{4} \epsilon_{abcde} e^a \sigma^{bc} \sigma^{de} \right]$$

where the 3-form ϕ is defined as $\phi := \epsilon_{abcde} e^a \sigma^{bc} \gamma^{de}$. It is now evident that if $\sigma \neq 0$, this action is not linearly supersymmetric under $\delta\psi = \nabla \epsilon$.

The effective potential $U(\sigma)$ governing the values of σ is defined as

$$e^{i f(-U(\sigma))} = \int D\psi D\bar{\psi} e^{i I_W[\psi, \sigma]}$$

and we obtain

$$U(\sigma^{ab}) = -\frac{k}{4} \epsilon_{abcde} e^a \sigma^{bc} \sigma^{de} + i \log \det \left(2i \mathcal{R} \nabla - \frac{\phi}{2} \right).$$

In a semiclassical approximation, the value of σ is given by the minimum of the effective potential U . If this minimum is not zero, then supersymmetry is broken. We already see that in the limit of large k the solution is in fact $\sigma^{ab} = 0$. We now study the contribution from the first quantum correction.

Although it is nice to have general background independent formula for U , the actual computation of the determinant is complicated because the operator $\mathcal{R} \nabla$ is nonminimal and the standard heat kernel formulas cannot be applied in a straightforward way.

In order to get an idea into the structure of the determinant we consider the large l regime. In this regime, spacetime is approximately flat and we shall compute the explicit value of U in that case. Specifically we consider fields W which are slowly varying, while ψ is fast varying, as compared to l . In this regime we approximate

$$R^{ab} + e^a e^b \approx e^a e^b$$

$$D\psi + \frac{1}{2} e \psi \approx d\psi. \quad (9)$$

Rather than computing the effective potential directly from the above formula, it is convenient to rewrite the fermionic action in flat space. We first note that if the 2-form σ^{ab} can have nonzero values, Poincaré invariance dictates its general form,

$$\sigma^{ab} = \frac{m}{6} e^a \wedge e^b, \quad (10)$$

where m is a constant which can be interpreted as the fermion mass. The action $I_W[\psi, \sigma]$ reduced to flat space becomes

$$I[\psi] = \int \left[\bar{\psi}_\mu (\gamma^{\mu\rho\sigma} \partial_\rho - m \gamma^{\mu\sigma}) \psi_\sigma - \frac{5}{6} k m^2 \right].$$

Of course, if $m=0$ $I[\psi]$ is linearly supersymmetric under $\delta\psi_\mu = \partial_\mu \epsilon$, and confirms that this action is the flat space analogous to $I_W[\psi, \sigma]$.

The question we would like to ask is whether or not the true minimum of the effective potential is $m=0$.

The determinant can now be computed by direct calculation. We first compute the determinant of the Dirac matrices and obtain

$$\det(\gamma^{\mu\sigma\rho}\partial_\rho - m\gamma^{\mu\sigma}) = 2^8 m^8 (m^2 + \partial^\mu\partial_\mu)^6. \quad (11)$$

The pole at $m=0$ is expected because the action is gauge invariant at this point, and the integration over fermions would require gauge fixing. This pole is however canceled by the determinant of the second class constraints present in the fermionic action. Let us pause to explain this point.

In a theory with second class constraints $G_A \approx 0$ with $\det\{G_A, G_B\} \neq 0$, the functional measure contains the factor $\det^{\pm 1/2}\{G_A, G_B\}$ where the plus/minus sign corresponds to boson or fermions (see [15]).

The flat space fermionic action in a 4+1 decomposition reads

$$\begin{aligned} \mathcal{L} = & -\psi_i^\dagger \gamma^{ij} \dot{\psi}_j + \bar{\psi}_i (\gamma^{ijk} \partial_j - m \gamma^{ik}) \psi_k - \frac{5}{6} k m^2 \\ & - \psi_0^\dagger (\gamma^{ij} \partial_j - m \gamma^i) \psi_i + \psi_i^\dagger (-\gamma^{ij} \tilde{\partial}_j - m \gamma^j) \psi_0. \end{aligned}$$

From here we identify the Poisson bracket

$$\{\psi_i^\alpha(x), \psi_j^\dagger{}_\beta(y)\} = (\hat{\gamma}_{ij})^\alpha{}_\beta \delta^{(4)}(x, y) \quad (12)$$

where $\hat{\gamma}_{ij} \gamma^{jk} = \delta_i^k$, and the two constraints,

$$G := (\gamma^{ij} \partial_j - m \gamma^i) \psi_i, \quad G^\dagger := \psi_i^\dagger (-\gamma^{ij} \tilde{\partial}_j - m \gamma^j) \quad (13)$$

whose Poisson bracket

$$\{G(x), G^\dagger(y)\} = \frac{4}{3} m^2 \delta^{(4)}(x, y),$$

is invertible, as claimed. These constraints become first class in the limit $m \rightarrow 0$ and generate the supersymmetry transformations. For $m \neq 0$ they are second class.

Since both G and $G^\dagger$ carry spinor indices, there are in total 8 constraints G_A ($A=1 \dots 8$), and since their Poisson bracket scales as m^2 we have

$$\det\{G_A, G_B\}^{-1/2} \sim m^{-8}.$$

This factor in the fermion measure cancels the pole in Eq. (11) exactly. This cancellation does not mean that we can extrapolate our results all the way to $m=0$. It only means that our calculation is correct no matter how small m is; at the exact value $m \equiv 0$ the whole theory is different, gauge fixing is necessary, and (finite) discontinuities are actually expected to arise [16].

The final expression for the effective potential in momentum representation and Euclidean space is then

$$U(m) = \frac{5k}{6} m^2 - 6 \int_{\Lambda} \frac{d^5 p}{(2\pi)^5} \log\left(1 + \frac{m^2}{p^2}\right),$$

where we have chosen $U(0)=0$ and Λ is an $SO(5)$ -invariant UV cutoff. Before going any further, let us recall that the fermionic quartic interaction we have considered here is not renormalizable in five dimensions. As a consequence, expanding the potential $U(m)$ into positive powers of Λ one finds a term $m^4 \Lambda$ that cannot be absorbed by a

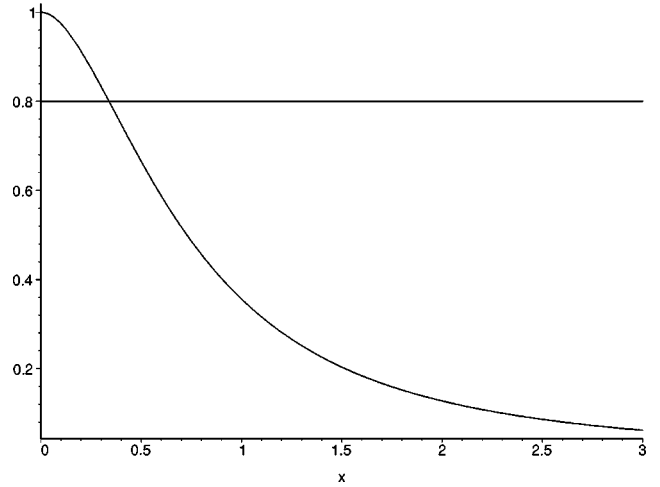


FIG. 1. Graphical analysis of Eq. (16). The horizontal lines represent different values for k'^3/Λ^3 . Nonzero solutions (intersections) exist only for $k'^3/\Lambda^3 < 1$. The value of m at which the intersection occurs increases as $k'^3/\Lambda^3 \rightarrow 0$.

redefinition of the coupling k . We shall proceed by identifying the physical parameters of the theory and write the cutoff in terms of them. Not unexpectedly, we find that Λ must be of the order of Planck's energy.

The potential $U(m)$ is clearly stable since, for a given cutoff, $U(m \rightarrow \pm\infty) \rightarrow +\infty$, and there are no other poles. The next question is whether symmetry breaking takes place. Taking the derivative of U with respect to m one finds the equation

$$\frac{5km}{3} = \int_{\Lambda} \frac{d^5 p}{(2\pi)^5} \frac{12m}{m^2 + p^2}, \quad (14)$$

describing the minima of U . One solution to this equation is $m=0$ corresponding to the unbroken phase. We would like to know if there are other solutions with $m \neq 0$, and less energy. In the following discussion it will be convenient to redefine the Chern-Simons coupling k as

$$k = \frac{k'^3}{5\pi^3}. \quad (15)$$

As it happens for all dimensions greater than two [12,17], Eq. (14) has a two-phase structure depending on the values of k'/Λ . In fact evaluating the integral in Eq. (14) we see that its nonzero solutions must satisfy

$$\frac{k'^3}{\Lambda^3} = 1 - 3 \frac{m^2}{\Lambda^2} + 3 \frac{m^3}{\Lambda^3} \arctan \frac{\Lambda}{m}. \quad (16)$$

This equation is plotted in Fig. 1. The left hand side is represented by straight lines, and the right hand side by the curve starting at 1, for $m=0$, and going monotonically to zero as $m \rightarrow \infty$. The intersections define nonzero solutions. The graph is symmetrical under $m \rightarrow -m$.

We first note that the intersection occurs only if

$$\frac{k'}{\Lambda} < 1. \quad (17)$$

If this condition is not satisfied, the only solution to Eq. (14) is $m=0$, and the symmetry is not broken.

From Eqs. (15) and (5) we see that $k' = 5^{1/3} \pi E_p$ where E_p is Planck's energy. Since the cutoff is the largest energy scale we conclude that the physical regime is in fact the broken phase. Equation (16) is satisfied, and nonzero solutions for m exist. We shall see, however, that Λ should not be too big.

Assume then that a nonzero solution to Eq. (16) exists, and let us call it M . We identify M with the physical mass of the fermions. On physical grounds we expect $M \ll \Lambda$, and hence we focus on the region near the origin of Fig. 1. In this region we can approximate Eq. (16) and M satisfies

$$k'^3 = \Lambda^3 - 3\Lambda M^2. \quad (18)$$

Now, since M is the physical mass “fixed by experiments” it is natural to use this equation to eliminate the cutoff in terms of the two physical quantities M and k' . To first order in M^2/k'^2 we find

$$\Lambda = k' \left(1 + \frac{M^2}{k'^2} + \dots \right). \quad (19)$$

Now we replace this value of the cutoff in the original potential and find the leading contribution in the large k' expansion

$$U(m) = \frac{k'}{4\pi^3} (m^4 - 2M^2 m^2) + \mathcal{O}(1). \quad (20)$$

This potential does not depend on the cutoff and exhibit symmetry breaking for all positive values of M^2 . By construction the value of m at the minimum is M , and

$$U(M) = -\frac{k'}{4\pi^3} M^4 < U(0).$$

Hence $m=M$ and not $m=0$ represents the true ground state.

It is interesting now to observe that $\Lambda \sim k' \sim k^{1/3}$ and, in view of Eq. (5), we find the expected result $\Lambda \sim E_p$ (plus small corrections given by the fermions mass). Equations (16) and (5) tell us that the fermion mass will satisfy $m \ll \Lambda$, provided Λ is of the order of E_p . If the cutoff is taken all the way to infinity then m diverges as well.

We would like to end with some comments and future prospects. We have shown that the effective potential governing the vacuum expectation value of the operator $\sigma^{ab} = (1/k) \langle \bar{\psi} \gamma^{ab} \psi \rangle$ does exhibit symmetry breaking. We have only computed the value for the potential on a flat background: it would be very interesting to find U for an arbitrary background. This would allow the study of the back reaction from the fermions fields to the geometry, and would be particularly relevant in view of the results of [7]. In fact the original motivation for this calculation was to study the possibility of nontrivial vacuum sources in Eq. (2) coming from the fermions, but in order to analyze this point properly we need the effective potential on a general field W . Finally, it would also be interesting to study the effect of the interaction term $(1/k) \bar{\psi} \psi \bar{\psi} D \psi$ that we have discarded in the large k limit.

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